

Macroscopic Screening of Coulomb Potentials From UV/IR-Mixing

Robert C. Helling^{ab}, Jiangyang You^b

^a*Ludwig-Maximilians-Universität München
Theresienstraße 37
80333 München, Germany*

^b*School of Engineering and Science
Jacobs University
Postfach 750561
28725 Bremen, Germany*

E-mail: helling@atdotde.de, j.you@iu-bremen.de

ABSTRACT: We compute the static potential in a non-commutative theory including a term due to UV/IR-mixing. As a result, the potential decays exponentially fast with distance rather than like a power law Coulomb type potential due to the exchange of massless particles. This shows that when quantum effects are taken into account the introduction of non-commutativity not only modifies physics at short distances but has dramatic macroscopic consequences as well. As a result, we give a *lower* bound on the scale of non-commutativity (if present at all) to be compatible with observations.

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1. Introduction

We have two extremely successful approaches to understand the fundamental interactions: Quantum field theory exemplified in the standard model of particle physics and general relativity governing the gravitational forces dominating the universe at large scales. It is widely believed that for a unification of these two approaches one has to develop a description of a quantum space-time which has to approximate the structure of a differentiable manifold at distances larger than the Planck scale.

Probably the most conservative approach to such a quantum space-time is to promote the coordinates to quantum operators which no longer commute and thereby induce a fundamental quantum uncertainty of geometry at the Planck scale[1][2] . This scenario gained a lot of momentum a few years ago when it became clear that it arises naturally from string theory in backgrounds with non-vanishing Kalb-Ramond 2-form $B_{\mu\nu}$ [3, 4, 5] .

The simplest version of space-time non-commutativity is obtained from commutation relations

$$[x^\mu, x^\nu] = i\theta^{\mu\nu}$$

for a constant, anti-symmetric matrix θ . It leads to the Moyal-Weyl *-product for fields living in this space time

$$(f * g)(x) = e^{\frac{i}{2}\theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \frac{\partial}{\partial z^\nu}} f(y)g(z) \Big|_{x=y=z}$$

By replacing the ordinary commutative product between fields in the action many field theories can adapted to this non-commutative space-time. For the standard model, this has been done in [6]. In [7] effective action of the non-commutative

Wess-Zumino was considered; as a result of supersymmetry this does not lead to the UV/IR-mixing terms that we study in this paper.

In these theories, θ plays the same role as \hbar in quantum mechanics. In quantum mechanics, the correspondence principle states that the quantum deformation of the classical theory is only noticeable at small scales where variations of the classical action are of the order of \hbar . Similarly, one could hope that a space-time non-commutativity with the entries of θ being of the order of ℓ_P^2 would only be relevant for physics at Planckian distances or with typical momenta of the order of the Planck mass while macroscopic physics would be indistinguishable from a theory on a commutative space.

This expectation was shaken by [8] where loop corrections to non-commutative scalar field theory were computed. There it was found that while planar diagrams receive no modifications due to non-commutativity, loop momentum integrals of non-planar diagrams have additional phase factors which generically render integrals which are UV divergent in the commutative theory finite. Thus the non-commutativity acts as a UV regulator.

For special external momenta, and most importantly $p \rightarrow 0$, the regulator is not effective and the UV divergence reappears now in the form of a IR divergence. Quadratic divergences of the commutative theory thus lead to terms of the form

$$\phi \frac{1}{p \circ p} \phi = \phi \frac{1}{p^\mu \theta_{\mu\nu} \theta^{\nu\rho} p_\rho} \phi \quad (1.1)$$

in the effective action. These terms notably are not of the form of the *-product interactions of the classical theory and it seems likely that at each loop order new counter terms are needed. This would render the non-commutative theories non-renormalisable.

This expectation turned out to be too pessimistic as in [9, 10, 11] it was shown that ϕ^4 theory on non-commutative \mathbb{R}^4 is in fact renormalisable. There it was shown that only a single further counter-term is required. In those papers, for technical reasons this counter-term was taken to be

$$x^\mu * \phi * \phi * x_\mu \quad (1.2)$$

to make the action symmetric under Fourier transform. It cancels the divergence of the form (1.1) as it has the same scaling dimension and $\theta_{\nu\mu} x^\nu *$ acts like ∂_μ . Thus, up to finite terms, it is highly plausible[12] that including a counter term of the form of (1.1) also yields a renormalisable ϕ^4 theory although an actual proof seems to be much more involved and we do not attempt it here. For the case of self-dual theta this theory has in the meantime been shown to be renormalisable[13]. This motivates the approach we will take in this paper: The term (1.1) is not just the result of a one loop calculation but it should be thought of as part of the quadratic part of the effective action which if not present right from the start is generated

by quantum corrections. Its contribution can be resummed by instead of the usual Feynman propagator using the functional inverse of $\square + \frac{1}{p \circ p}$ as the propagator.

Thus we take our theory to be given by

$$\mathcal{S} = \int d^4x \left(\partial^\mu \phi \partial_\mu \phi + \phi \frac{1}{p \circ p} \phi + \frac{\lambda}{4!} \phi * \phi * \phi * \phi \right) \quad (1.3)$$

where as usual $p_\mu = i\partial_\mu$. Note well that this new term is non-perturbative in θ and becomes singular in the commutative limit. For simplicity of notation we have not included a mass term as this does not alter the conclusions of our argument. In addition, we could have included a coupling constant in front of the new term but as in the following we are dealing just with the quadratic part of the theory (and will not need the ϕ^4 -interaction once it generated the new term), such a coupling constant can always be absorbed in the definition of θ .

The choice of considering a scalar field is merely for clarity of presentation. The discussion of this paper directly generalizes to any other field theory with UV/IR-mixing. This justifies our misnomer ‘‘Coulomb potential’’ for a $1/r$ potential transmitted by a massless scalar instead of a vector field.

In the existing literature, an effective action including this one loop induced term has been discussed as giving a potential in momentum space which has its minimum away from $p_\mu = 0$ thus leading to a spontaneous breaking of translation invariance and the formation of stripes[14, 15, 16, 17]. Lattice calculations have confirmed that this effect persists non-perturbatively[18, 19] (and [20, 21] in the case of the fuzzy sphere).

In this paper however, we take a different point of view and treat all terms quadratic in the field as part of the free theory determining the propagator. We will find that instead of the power law behavior of the propagator as $1/p^{D-2}$ in the commutative theory the non-commutative propagator is decaying exponentially fast. Consequently, the force the field is transmitting is short ranged and effectively cut-off at the scale of the non-commutativity. We find that even a small amount of non-commutativity has drastic effects at macroscopic scales as it screens long range forces. Our conclusion will thus be that it is not correct to think of non-commutativity modifying a theory only at short scales — possibly the Planck scale. As soon as quantum effects which result in UV/IR mixing are considered, the theory is drastically changed at scales large compared to the non-commutativity.

2. Screened Coulomb interaction

For concreteness, let us work in 3+1 dimensional Minkowski space. We want to compute the potential due to a source

$$J(x) = \int d\tau \delta^{(4)}(x - r(\tau)).$$

Let us assume the source is static and the force is mediated by a field with momentum space propagator $G(p) = G(E, \vec{p})$. Then the potential is given by

$$V(t, \vec{x}) = \int d^4 x' \int d^4 p G(p) e^{ip(x-x')} J(x') = \int d^3 \vec{p} G(0, \vec{p}) e^{i\vec{p} \cdot (\vec{x} - \vec{r})}.$$

Thus, in order to obtain the potential we have to compute the spatial Fourier transform of the propagator at $E = 0$.

For the scalar theory with the action (1.3), the momentum space propagator is given by

$$G(p) = \frac{1}{p^2 + \frac{1}{p \circ p}}.$$

Any finite coefficient of the $p \circ p$ -term in the action can be absorbed in the definition of $\theta^{\mu\nu}$ as in this paper we will only be concerned with the propagator and the only appearance of $\theta^{\mu\nu}$ is quadratically in \circ .

As a warm up, we will consider a Euclidean theory with self-dual

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta & 0 & 0 \\ -\theta & 0 & 0 & 0 \\ 0 & 0 & 0 & \theta \\ 0 & 0 & -\theta & 0 \end{pmatrix},$$

as in this case $p \circ p = \theta^2 p^2$ and the propagator only depends on p^2 . We can easily evaluate the static potential in polar coordinates:

$$V(r) = \int d^3 \vec{p} \frac{e^{i\vec{p} \cdot \vec{r}}}{\vec{p}^2 + \frac{1}{\theta^2 p^2}} = 2\pi^2 \frac{e^{-\frac{r}{\sqrt{2|\theta|}}} \cos\left(\frac{r}{\sqrt{2|\theta|}}\right)}{r} \quad (2.1)$$

The effect of the non-commutativity is to dress the $1/r$ Coulomb potential with an oscillatory factor and an exponential decay over the scale of the non-commutativity. The power law character of the force law has been replaced by a finite interaction range governed by the length scale of the non-commutativity. Indeed, the deviation from the Coulomb behavior is most pronounced at macroscopic distances $r \gg \sqrt{|\theta|}$ and we have to conclude that it is incorrect to think of the non-commutativity as a correction affecting physics only at very short distances.

It might come as a surprise that the commutative $1/r$ potential is not recovered as a smooth $\theta \rightarrow 0$ limit but in the opposite $\theta \rightarrow \infty$ limit. This however is due to the nature of the UV/IR-mixing term which blows up for small θ .

Our result demonstrates that it is essential to fully resum the contribution of all orders in θ in order not to miss the effects of the non-locality of the $*$ -product. The effect we study in this paper would have been missed in any investigation which had expanded the theory in θ and then had truncated the infinite series after a finite number of terms.

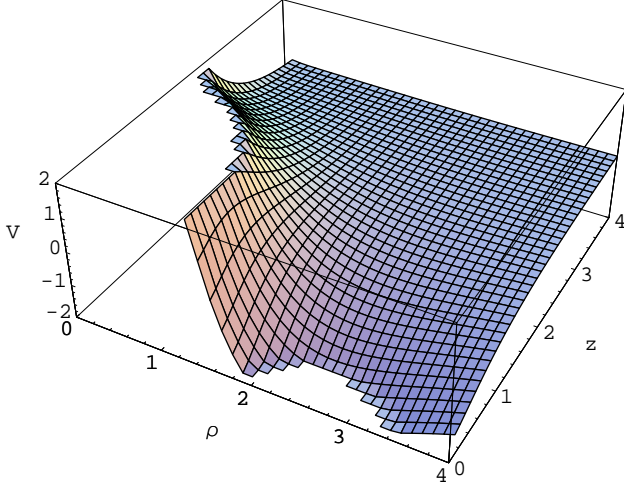


Figure 1: The static potential, coordinates in units of $\sqrt{\theta}$.

Let us now consider a more general non-commutativity. We want to consider a situation with (possibly after an appropriate boost) only spatial non-commutativity. We can then always pick coordinates such that

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \theta & 0 \\ 0 & -\theta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

the non-commutativity is only in the xy -plane and $\theta > 0$. In cylindrical coordinates, after a change of coordinates $p \mapsto p/\sqrt{\theta}$, we have to evaluate

$$V(\varrho, z) = \frac{1}{\sqrt{\theta}} \int_0^\infty p_r dp_r \int_0^{2\pi} dp_\phi \int_{-\infty}^\infty dp_z \frac{e^{i(p_r \varrho \cos p_\phi + p_z z)/\sqrt{\theta}}}{p_z^2 + p_r^2 + \frac{1}{p_r^2}}$$

Figure 1 shows this static potential evaluated numerically. One can see it is indeed decaying exponentially fast.

The p_ϕ and p_z integrals are easily evaluated and we find

$$V(\varrho, z) = \frac{4\pi^2}{\sqrt{\theta}} \int_0^\infty dp_r p_r \frac{e^{-|z|\sqrt{p_r^2 + \frac{1}{p_r^2}}/\sqrt{\theta}} J_0(p_r \varrho/\sqrt{\theta})}{\sqrt{p_r^2 + \frac{1}{p_r^2}}}.$$

We cannot evaluate this remaining integral analytically. However, it is sufficient to determine it asymptotically for distances $\sqrt{\varrho^2 + z^2}$ large compared to $\sqrt{\theta}$ using the method of steepest descent. We fix the direction $c = \varrho/|z|$ use the asymptotic form of the Bessel function

$$J_0(x) \rightarrow \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4).$$

Eventually

$$V(cz, z) \rightarrow \sqrt{\frac{32\pi^3}{\sqrt{\theta}c|z|}} \int_0^\infty dp_r \sqrt{\frac{p_r}{p_r^2 + \frac{1}{p_r^2}}} e^{-|z|\sqrt{p_r^2 + \frac{1}{p_r^2}}/\sqrt{\theta}} \cos\left(\frac{p_r c|z|}{\sqrt{\theta}} - \frac{\pi}{4}\right). \quad (2.2)$$

Next, we rewrite the cosine as a sum of two complex exponentials. For $|z| \gg \sqrt{\theta}$ the integral is dominated by the contribution from the points where the phase $\Phi(p_r) =$

$\sqrt{p_r^2 + \frac{1}{p_r^2}} + i\alpha p_r$ is stationary (with $\alpha = \pm 1$). For each α there are contributions from four stationary points $p_{r,i}$ (which are a function of c only but not of z or θ) obeying

$$p_{r,i} - \frac{1}{p_{r,i}^3} = -i\alpha c \sqrt{p_{r,i}^2 + \frac{1}{p_{r,i}^2}}. \quad (2.3)$$

By evaluating the p_r integral to leading order in $z/\sqrt{\theta}$ we find

$$V(z, cz) = \sum_{\pm, i} \frac{4\pi^2}{|z|} \left(1 + \frac{1}{p_{r,i}^4}\right)^{\frac{1}{4}} \frac{\pm p_{r,i}^4}{\sqrt{c(2 + 6p_{r,i}^4)}} e^{\pm(\Phi(p_{r,i})|z|/\sqrt{\theta}) - \pi i/4} \left(1 + O\left(\frac{1}{|z|}\right)\right) \quad (2.4)$$

as explained in the appendix. As detailed there, the sum has to be taken over all solutions $p_{r,i}$ of

$$\frac{1}{p_{r,i}^4} = 1 - \frac{c^2 \pm c\sqrt{c^2 - 8}}{2}. \quad (2.5)$$

By analyzing the integrand of (2.2) for large and small p_r it can be seen that $V(cz, z) \rightarrow 0$ for large z . Thus we can conclude that the sign in the exponential in (2.4) will always be such that the real part of the coefficient of $|z|$ is positive. In the appendix, it is shown that this is indeed the case.

The expression (2.4) constitutes the main result of this work. It shows that for distances z large compared to the length scale of non-commutativity $\sqrt{\theta}$, the interaction which is transmitted by the field ϕ dies off exponentially fast in contrast to the commutative situation where it only falls off like $1/z$. The effect is as if due to the non-commutativity the field ϕ had acquired a direction dependent mass of the order of $1/\sqrt{\theta}$.

3. Conclusions

In this paper, we investigated the effect of non-commutativity on the long range interactions mediated by a scalar field taking into account UV/IR mixing. This effect due to loop diagrams induces new quadratic terms in the effective action with negative powers of θ , the scale of non-commutativity.

We showed that by including these terms into a resummed propagator the power law effective potential is modified to a Yukawa type potential with exponential fall-off with a direction dependent decay length of the order of $\sqrt{\theta}$.

This calculation shows explicitly that the assumption motivated by a classical analysis that the deformation leading to non-commutativity of space-time coordinates only modifies physics at short distances (typically the Planck scale) is not true once quantum effects are taken into account. Rather, the drastic consequences of the non-commutativity is that it switches off the interaction for macroscopic scales.

This raises the question of viability of simple non-commutative models for phenomenology given that at macroscopic scales we do observe long range power-law interactions but no signs of non-commutativity. In this paper we have only treated a single scalar field but the argument should trivially generalize to fields of higher spin as soon as the theory has UV/IR-mixing. The specific term $1/p \circ p$ arises from integrals which are quadratically divergent in the commutative theory. Thus the argument should directly apply for example for the Higgs self-energy in the standard model giving a lower bound on $\sqrt{\theta}$ at the order of the inverse Higgs mass. Terms with milder divergence lead to slightly different terms in the effective action which however will have a similar behavior that they become large for small $\sqrt{\theta}$ since they arise from divergences which are regularized by the non-commutativity resulting in similar macroscopic consequences of the non-commutativity and resulting in similar bounds.

Thus it is likely a generic situation that quantum effects transmit non-localities due to non-commutativity which classically appear at short scales to arbitrary large scales. This should be seen as a challenge to model-building based on non-commutative space-time.

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A. The effective potential is decaying exponentially

In this appendix, it will be shown that the real part of the phase $\Phi(p_{r,i})$ at the stationary points is positive and thus the effective potential in (2.4) is indeed decaying at large distances.

We want to determine the sign of the real part of

$$\Phi(p_{r,i}) = \sqrt{p_{r,i}^2 + \frac{1}{p_{r,i}^2}} + i\alpha c p_{r,i}$$

at its stationary points with $\alpha = \pm 1$ and $c \geq 0$. As we pick the branch-cut to be along the negative real axis, all square-roots will have non-negative real parts. Thus it is sufficient to show that at the critical points we have $\alpha \operatorname{Im}(p_{r,i}) \leq 0$.

Setting the derivative of Φ to zero yields

$$\alpha \left(p_{r,i} - \frac{1}{p_{r,i}^3} \right) = -ic \sqrt{p_{r,i}^2 + \frac{1}{p_{r,i}^2}}. \quad (\text{A.1})$$

Squaring this relation gives a quadratic equation for $1/p_{r,i}^4$ which is solved by

$$\frac{1}{p_{r,i}^4} = 1 - \frac{c}{2} \left(c + \alpha\beta\sqrt{c^2 - 8} \right) \quad (\text{A.2})$$

with another sign $\beta = \pm 1$. However, as to obtain this solution we have squared (A.1), only four of the eight solutions for $p_{r,i}$ in (A.2) will really solve (A.1). However, the other solutions will solve (A.1) for the other choice of α . Therefore, instead of fixing α and determining which four of the eight $p_{r,i}$ is really a solution, we find the appropriate α_i for each of the four $p_{r,i}$.

Let us first consider the case $c^2 \geq 8$. From (A.1) and the non-negative real part of square-roots we find that the imaginary part of $\alpha_i p_{r,i}(1 - 1/p_{r,i}^4)$ is negative. But (A.2) implies that $1 - 1/p_{r,i}^4$ is real and positive. Thus the imaginary part of $\alpha_i p_{r,i}$ is negative and indeed $\exp(-|z|\Phi(p_{r,i})/\sqrt{\theta})$ decays for large $|z|$.

It remains to consider $c^2 < 8$. As with any solution $p_{r,i}$ of (A.2) also $i^n p_{r,i}$ are solutions, we can first consider the solution such that $|\arg(p_{r,1})| \leq \pi/4$. In this case, it is easy to see that the imaginary part of $p_{r,1} - 1/p_{r,1}^3$ has the same sign as the imaginary part of $p_{r,1}$. Again using (A.1) and the non-negative real part of square-roots implies that $\text{Im}(\alpha_1 p_{r,1})$ is not positive. Substituting the next solution $p_{r,2} = -p_{r,1}$ in (A.1), the RHS does not change and we have $\alpha_2 = -\alpha_1$ to compensate the change of sign of $p_{r,i} + 1/p_{r,i}^3$. Thus $\alpha_1 p_{r,1} = \alpha_2 p_{r,2}$.

Next, consider $p_{r,3} = ip_{r,1}$ which has positive imaginary part. For any complex number $z = x + iy$ one has

$$\text{Im} \left(z^2 + \frac{1}{z^2} \right) = 2xy \left(1 - \frac{1}{|z|^4} \right)$$

From the explicit solution we have $|1/p_{r,i}^4| = 1 + c^2$. Thus $\text{Im}(p_{r,1}^2 + 1/p_{r,1}^2)$ has the opposite sign of $\text{Im}(p_{r,1})$. Using this together with $\sqrt{-z} = -i\sqrt{z} \text{sgn}(\text{Im}(z))$ in evaluating the RHS of (A.1) for $p_{r,3}$ gives $\alpha_3 = -1$. Thus $\text{Im}(\alpha_3 p_{r,3}) \leq 0$ as well. The same is true for $p_{r,4} = -p_{r,3}$ as can be shown using the same argument as in the case of $p_{r,2}$.

This concludes our proof that for all stationary points $p_{r,i}$ of Φ the sign α_i happens to adjust to make $\text{Im}(\alpha_i p_{r,i}) \leq 0$ and thus the effective potential decays at large distances.

References

- [1] H. S. Snyder, *Quantized space-time*, *Phys. Rev.* **71** (1947) 38–41.
- [2] S. Doplicher, K. Fredenhagen, and J. E. Roberts, *The quantum structure of space-time at the planck scale and quantum fields*, *Commun. Math. Phys.* **172** (1995) 187–220, [[hep-th/0303037](#)].

- [3] M. R. Douglas and C. M. Hull, *D-branes and the noncommutative torus*, *JHEP* **02** (1998) 008, [[hep-th/9711165](#)].
- [4] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *JHEP* **09** (1999) 032, [[hep-th/9908142](#)].
- [5] M. R. Douglas and N. A. Nekrasov, *Noncommutative field theory*, *Rev. Mod. Phys.* **73** (2001) 977–1029, [[hep-th/0106048](#)].
- [6] X. Calmet, B. Jurco, P. Schupp, J. Wess, and M. Wohlgenannt, *The standard model on non-commutative space-time*, *Eur. Phys. J.* **C23** (2002) 363–376, [[hep-ph/0111115](#)].
- [7] H. O. Girotti, M. Gomes, A. Y. Petrov, V. O. Rivelles, and A. J. da Silva, *The low energy limit of the noncommutative wess-zumino model*, *JHEP* **05** (2002) 040, [[hep-th/0101159](#)].
- [8] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, *Noncommutative perturbative dynamics*, *JHEP* **02** (2000) 020, [[hep-th/9912072](#)].
- [9] H. Grosse and R. Wulkenhaar, *Renormalisation of ϕ^4 -theory on non-commutative \mathbb{R}^4 to all orders*, *Lett. Math. Phys.* **71** (2005) 13–26.
- [10] H. Grosse and R. Wulkenhaar, *Renormalisation of ϕ^4 theory on noncommutative \mathbb{R}^4 to all orders*, [hep-th/0403232](#).
- [11] H. Grosse and R. Wulkenhaar, *Renormalisation of ϕ^4 theory on noncommutative \mathbb{R}^4 in the matrix base*, *Commun. Math. Phys.* **256** (2005) 305–374, [[hep-th/0401128](#)].
- [12] H. Grosse. private communication.
- [13] R. Gurau, J. Magnen, V. Rivasseau, and A. Tanasa, *A translation-invariant renormalizable non-commutative scalar model*, [arXiv:0802.0791](#) [[math-ph](#)].
- [14] M. Van Raamsdonk, *The meaning of infrared singularities in noncommutative gauge theories*, *JHEP* **11** (2001) 006, [[hep-th/0110093](#)].
- [15] A. Armoni and E. Lopez, *Uv/ir mixing via closed strings and tachyonic instabilities*, *Nucl. Phys.* **B632** (2002) 240–256, [[hep-th/0110113](#)].
- [16] Z. Guralnik, R. C. Helling, K. Landsteiner, and E. Lopez, *Perturbative instabilities on the non-commutative torus, morita duality and twisted boundary conditions*, *JHEP* **05** (2002) 025, [[hep-th/0204037](#)].
- [17] P. Castorina and D. Zappala, *Nonuniform symmetry breaking in noncommutative $\lambda \phi^4$ theory*, *Phys. Rev.* **D68** (2003) 065008, [[hep-th/0303030](#)].

- [18] W. Bietenholz, F. Hofheinz, and J. Nishimura, *Numerical results on the non-commutative $\lambda \phi^4$ model*, *Nucl. Phys. Proc. Suppl.* **129** (2004) 865–867, [[hep-th/0309182](#)].
- [19] W. Bietenholz, J. Nishimura, Y. Susaki, and J. Volkholz, *A non-perturbative study of 4d $u(1)$ non-commutative gauge theory: The fate of one-loop instability*, *JHEP* **10** (2006) 042, [[hep-th/0608072](#)].
- [20] M. Panero, *Quantum field theory in a non-commutative space: Theoretical predictions and numerical results on the fuzzy sphere*, *SIGMA* **2** (2006) 081, [[hep-th/0609205](#)].
- [21] M. Panero, *Numerical simulations of a non-commutative theory: The scalar model on the fuzzy sphere*, *JHEP* **05** (2007) 082, [[hep-th/0608202](#)].